Cantor and the Attempt to Refute Aristotle:

Infinite Sets and the Principle of Plenitude[1]

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This is the third "digital extension" to *Aristotle's "Not to Fear" Proof for the Necessary Eternality of the Universe* (New York: ExistencePS Press, 2019).

It is extremely rare to see a discussion of Cantor and Aristotle in the context of theology, the primary reason of course being that Cantor is known usually as a mathematician.

Yet Anne Newstead provides a fascinating account of how Cantor not only uses God and the Principle of Plenitude to justify his rejection of classical theories of the infinite but to argue on metaphysical grounds against Aristotle's view that the infinite cannot be actual in the relevant sense.[2] Cantor claims instead that the infinite *can* be actual, that some infinities are indeed actual, and that transfinite sets exist, leading to a host of modern paradoxes in mathematics and to the belief of some that one can only accept Cantor's views on faith.

I repeat the important passages in Newstead's article all at once, interjecting footnotes until the end, when I note the relevance of her insights for *Aristotle's "Not to Fear" Proof*.

As Newstead writes:

> Cantorian set theory requires a commitment to the existence of actual infinities. There are (according to Cantorian set theory) actually infinite sets of various sizes, each of a certain transfinite cardinality.
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> The 'infinite', once thought to be beyond the scope of mathematics, could be tamed. Cantorian set theory is revolutionary in many ways: philosophically and mathematically. ***It throws off the Aristotelian prohibition against actual***

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[2] Anne Newstead, "Intertwining metaphysics and mathematics: the development of Georg Cantor's set theory 1871-1887," from Academia.edu; no date or additional publication information given. Newstead, who has impressive credentials (*DPhil,* Oxford 2004, and part-time tutor at the University of Western Sydney), lists her affiliation in the article with School of Mathematics and Statistics, University of New South Wales, NSW 2052, Australia.

infinities in mathematics. [3] It provides a new way of conceiving of the structures of mathematics as sets. Set theory is powerful enough to embed hitherto existing mathematical theory within itself, thereby providing a foundation for mathematics. The Cartesian dream of a certain foundation for knowledge seems within reach, *if only the axioms and premises of set theory could be known with certainty*. Of course we know from subsequent history that the Cartesian dream was shattered. The assumptions of set theory—particularly the assumption that the infinite sets are well-ordered—have not met with universal agreement. Possibly, the best possible way of vindicating these axioms is 'post hoc': they produce powerful, fruitful mathematics and are consistent as far as we know, ***so they should be assumed to be true*** (p. 2; my emphases).[4]

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Prior to 1886 Cantor adopted a metaphysical system that may be described as both neo-Platonist and Spinozist in inspiration. The key principle of this metaphysical system is that God produces everything possible, i.e. everything 'that can fall under the scope of an infinite intellect'. In a letter to Eberhard Illigens of 21 May 1886, for example, Cantor argues in effect that, given God's omnipotence, every possible (consistent internally) [sic] can exist and become real:

> If I have recognised the inner consistency of a concept which points to a being, then ***the idea of God's omnipotence impels me to think of the being expressed by the concept as in some way actually realizable.***

On this basis, we may characterise Cantor's early metaphysics as subscribing to ***a version of the principle of plenitude: everything possible is actual*** (p. 6; my emphases).

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...he [Cantor] needed as loose a criterion for the reality of ideas as logical possibility to prove that his transfinite numbers were as real as finite numbers. Whereas initially he did think that his transfinite numbers were mere 'definite defined

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[3] Aristotle allows potential infinities in mathematics. It is often thought he allows no actual infinity of any kind, but this is disproven by the type of temporal infinite that he claims is like the "day" or the "Games" (leaving aside their finitude): one part goes away as another comes into existence (*Physics* III 6, 206a21-26). Eternal existence is not, nor could it be, merely potential; see *Aristotle's "Not to Fear" Proof*, pp. 28-9; 32; 34; 41; 52-3; 57-8; and 93.

[4] Powerful systems can be produced, like theories of a flat earth, that arguably are consistent. Some systems of logic are consistent but with absurd axioms. A powerful Gestapo-state could be consistent in its law and the brutal application of those laws to citizens so as to make for a very powerful government and army that controls a country and part of the world for years, if not generations. That hardly establishes the truth of the claim that this is the best type of state. Whether non-Cantorian mathematics can achieve equally "fruitful mathematics" now or in the future without the paradoxes of actual infinities, I leave to the mathematicians, and Newstead reports that there are many, and many philosophers, including M.A.E. Dummett, who are not persuaded by Cantor, of which more at the end.

infinity symbols', by the time of the *Grundlagen* (1883), he was convinced that they had as much right to be considered 'concrete numbers of real significance' as the finite numbers (p. 16; my italics).[5]

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Spinoza is traditionally regarded as a pantheist, because he identifies God with the whole of nature, and its immanent cause (*Ethics* I p18). Cantor is clearly not a pantheist in this sense, because he did not reject the idea of a transcendent God who stands outside of his creation. Nonetheless, as we have seen, at the time of the *Grundlagen*, he shared with Spinoza the principle that the immanent reality suffices for the transient reality of an idea. *This principle amounts to the collapse of the possible and the actual.* The coincidence of the possible and the actual is strictly independent of pantheism. However, once the divine essence is conceived of as the realm of all possibilities, the coincidence of possibility and actuality follows from the pantheist (and neo-Platonist) doctrine of the necessary emanation of things from the divine essence. *So pantheism, necessitarianism, and the modal principle of plenitude ('everything possible is actual') are a closely allied set of doctrines.*

Pantheism was both fashionable and heretical in mid-19th century Germany. In 1861, Pope Pius IX felt the need to issue a formal prohibition against pantheism, a sure sign that the doctrine was popular as well as threatening. Given the moral and intellectual support that Cantor received from Catholic neo-Scholastic theologians (such as Constantin Gutberlet) in the face of heavy criticism from mathematicians, it is likely that Cantor would have been motivated to avoid (at least the appearance of) heresy.[49] Cantor was aware that his opinions contradicted those of Thomas Aquinas, whose philosophy was fast becoming the official philosophy of the Church. *Some theologians feared his transfinite numbers might undermine the doctrine of creation by making a return to the notion of an eternal world possible.* As Cantor came to realise in the early months of 1886 through his correspondence with Cardinal Johannes Franzelin, by far the biggest challenge orthodoxy posed against his belief in the transfinite was to show that it did not lead to pantheism (p. 16; my italics).

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Unlike pantheists, Cantor does not think that God's infinity is exhausted in nature. However, one is hardly reassured by his enthusiastic endorsement of Leibniz's theory of monads, which is arguably just a consistent version of pantheism, overlaid with a few concessions to orthodox religion. One finds traces, too, of his allegiance to pantheism, in his use **of a kind of principle of plenitude**:

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[5] I assume that the "mere 'definite defined infinity symbols'" function like my so-called "drunken-eight," ∞, technically called the lemniscate, which is the symbol used to suggest that infinity is a definite quantity, similar to a number, when for Aristotle the symbol would really represent only a concept that always has something more outside of it and that therefore can be neither definite nor numerical in the strict sense; see *Aristotle's "Not to Fear" Proof*, pp. 44ff.

> One proof [of the reality of the Transfinite] proceeds from the concept of God and infers from the greatest perfection of God's essence *the possibility of the creation of a transfinite order*, from his supreme goodness **to the necessity that there should actually follow a Transfinite** (p. 19; my emphases).

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Cantor's next letter to Franzelin, dated 29 January 1886, contains a significant retreat from his previous position. He no longer tries to maintain the content of his philosophy while arguing that it separability [sic] from pantheism. Instead he modifies that system to meet the demands of orthodoxy. Cantor now claims in his letter that the creation of the finite as well as the transfinite realm has a merely 'subjective necessity for us' that follows from our contemplation of the divine nature:

> …it was not [upon further consideration] my opinion, to speak of an objective, metaphysical necessity of the creative act, which would have subjugated God's absolute freedom, but rather I wanted to point to a subjective necessity for us, that from God's absolute goodness and splendour there must result an actual creation, not merely of a finite order (not following from a part of God), but also of a transfinite order.

Cantor's notion of 'purely subjective necessity' could be a contortion designed to bend his views into orthodox shape. *But it is clear that he cannot have it both ways. Either his transfinite numbers are necessary, and God does not create freely, or the numbers are contingent beings and God creates freely.* In the end, Cantor's desire to protect what he saw as 'religious truth' won out over his desire to find a secure philosophical foundation for his transfinite numbers. In his later letters, he is careful to emphasize God's freedom, and cites Franzelin's final approval of his revised system repeatedly (pp. 20-1; my italics).

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…belief in an actual infinity realized apart from God has always been dangerous. Giordano Bruno was burned at the stake for believing in an infinity of worlds. His successor, Benedictus Spinoza was reviled as an atheist for two centuries after his work, his reputation following the opinion of his early critic, Pierre Bayle, in his influential *Dictionnaire historique et critique* (1697). With the birth of German romanticism, and the revelation that Lessing and Jacobi admired his work, Spinoza's reputation was rehabilitated. A Spinoza Renaissance flourished in Germany, and Spinoza's monism influenced the philosophies of Hegel, Schelling, and Goethe. It would not be surprising if this atmosphere contributed to Cantor's initial positive reception of Spinoza. But like Spinoza, *Cantor's belief in an infinity apart from a transcendent personal deity flew in the face of orthodox religion.* Mindful of the lessons of history, Cantor choose to walk a narrow path between orthodoxy and insincerity (p. 22; my italics).

The Ramifications for Aristotle

At least in Newstead's account, Cantor provides no *reductio ad absurdum* to counter Aristotle's view, similar to what Philoponus offers in his own attempt to undercut Aristotle (see pp. 36-66). That is, even though Aristotle denies that infinity is a number, Philoponus attempts to show that the Stagirite's own doctrine commits him either to treating it as one or to showing infinity can be smaller or larger or the like. Rather the justification for Cantor treating infinity not only as a number but as actual comes from purely theological, Christian doctrine, with the Principle of Plenitude embedded as part of the alleged demonstration, namely, what is possible in the mind of God is actual. In a way, this arguably follows the seventh version of the Principle of Plenitude that Aristotle clearly holds at *Physics* III 4, 203b30:

For eternal things, what may be, is.

as explained at: www.EPSpress.com/NTF/VariousVersionsOfThePrinciple.pdf

God is eternal and therefore what may be in his mind is. Of course for those who do not believe in (this type of Christian) God, the claim has no force.

The reader might object and say that the *Nous* (Mind) of *Metaphysics* Lambda that thinks of itself thinking might be always thinking of the infinities, which as a consequence are actual. The fatal problems with this position are already addressed in the section on *Nous* in Part 2 of *Aristotle's "Not to Fear" Proof* and in a previous digital extension "Does the Unmoved Mover Necessarily Imply the Physical Universe?" (available at www.EPSpress.com/NecessaryImplication.pdf). The primary problem is that any thinking being for Aristotle must have a brain, body and matter, which entails that it has both the potential to be *and not to be*. Thus, this *Nous* is contingent, which utterly contradicts the goal in Lambda to establish a necessarily eternal being that grounds the existence and motion of the universe.

Because my interest in Newstead's article, and Cantor's theory, is very narrow, only pertaining to how it affects Aristotle and my related book, I finish with statements by those who have engaged the various related issues more deeply and more broadly. As Newstead informs us:

> Among some professional mathematicians, especially those of a constructivist bent, the view persists even today that Cantorian set theory is a kind of religion.

The implication of this view of Cantorian mathematics is that its theorems must be taken on faith without argument (p. 3).
...
In an exceptional work of careful historical scholarship, Michael Hallett excavates the theological and metaphysical background underlying Cantor's mathematics. Hallett clearly recognizes the influence of Cantor's views on the subsequent development of set theory. However, Hallett is unconvinced as to the philosophical worth of Cantor's metaphysical views. He writes that:

> *In the end*, it is important to divorce the main content of Cantor's ideas... from their theological origins. This is partly because the appeal to God in the way Cantor often does is ultimately no great help. For example, to claim that certain infinite sets or certain infinite numbers exist because it is possible for God to conceive, and therefore to create, them, or that they exist because they are 'ideas in the divine intellect' tells us no more than the bald claim that they exist (pp. 3-4; Hallett's italics).

...In a series of books starting with *The Elements of Intuitionism*, Michael Dummett argues against realism of the Cantorian sort on the grounds of the epistemic limitations of mathematicians. He urges that the idea of a completed, actual infinity in mathematics is '**incoherent**'. Taking the liberal finitist view of Aristotle, Kant, and Brouwer, he argues:

> From an intuitionistic standpoint, the platonistic conception is the result of blatantly transferring, from the finite case to the infinite one, a picture appropriate only to the former. In making this transference, the platonist destroys the whole **essence of infinity**, **which lies in the conception of a structure which is always in growth,** precisely because the process of construction is never completed.

Ironically, ... Cantor was dismissed by some influential mathematicians for being too 'philosophical' in the pejorative sense of the word (i.e. speculative and lacking in rigour). Nonetheless, Cantor's introduction of transfinite arithmetic and set theory constitutes a major philosophical contribution. **It provides a coherent theory of actual infinity**, and it does so with a level of mathematical detail and rigour not seen before. **It thus answers the historical objection that there can be no coherent conception of an actual infinity** (p. 4; my boldfacing).

I myself add the final word, because much as I appreciate Newstead's illuminating article, I must disagree on the last point. For Aristotle (and I would submit for us), this is all not an issue of "coherence," but of truth, at least fundamentally. A description of a purple-haired monster the size of a galaxy or a performance of dramatic musical theater like *Oedipus* or a composition in epic form can be coherent and even a cause for anxiety for children, teens or adults reading, hearing or seeing the stories enacted, all of it entailing strong believability on their parts. The

coherence (and resulting believability) hardly makes the story, or the description, true. Yet *this* is the critical issue, namely, *whether the existence of actual infinities can be claimed to be true or not*, with truth being, as Aristotle says, "to say of what is, that it is (*Metaphysics* IV 7, 1011b25)."[6]

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[6] For anyone appealing to Alfred Tarski's denigration of Aristotle's notion of truth, see my critique of Tarski at pp. 129-30, footnote 62.